Linear Algebra I

24/01/2022, Monday, 16:00 – 18:00

You are **NOT** allowed to use any type of calculators.

---

1 Cramer's rule

5 + 15 = 20 pts

Let a, b, c, and d be distinct real numbers. Consider the following system of linear equations in the unknowns x, y, z

$$\begin{aligned} x + y + z &= 1 \\ ax + by + cz &= d \\ a^2x + b^2y + c^2z &= d^2. \end{aligned}$$

(a) Show that

$$\det\left(\begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{bmatrix}\right) = (a-b)(b-c)(c-a).$$

(b) By using Cramer's rule, show that

$$x = \frac{(d-b)(c-d)}{(a-b)(c-a)}, \quad y = \frac{(a-d)(d-c)}{(a-b)(b-c)}, \quad z = \frac{(b-d)(d-a)}{(b-c)(c-a)}.$$

2 Vector spaces

4 + 5 + (8 + 8) = 25 pts

Let $\boldsymbol{x} \in \mathbb{R}^n$ be a nonzero vector. Consider the set

$$S_{\boldsymbol{x}} = \{A \in \mathbb{R}^{n\times n} \mid A = A^T \text{ and } \boldsymbol{x} \text{ is an eigenvector of } A\}.$$

and the mapping $L_{\boldsymbol{x}} : S_{\boldsymbol{x}} \to \mathbb{R}^{n\times n}$ be given by

$$L_{\boldsymbol{x}}(A) = A\boldsymbol{x}\boldsymbol{x}^T + \boldsymbol{x}\boldsymbol{x}^T A.$$

(a) Show that $S_{\boldsymbol{x}}$ is a subspace of $\mathbb{R}^{n\times n}$.

(b) Show that $L_{\boldsymbol{x}}$ is a linear transformation and find $\ker L_{\boldsymbol{x}}$.

(c) Let $n = 2$ and $\boldsymbol{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

  (i) Find a basis for $S_{\boldsymbol{x}}$ and determine its dimension.

  (ii) Find a basis for $\ker L_{\boldsymbol{x}}$ and determine its dimension.

3 Least squares problem $10 + 10 = 20$ pts

Let

$$x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad x_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad x_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$y_1 = 1 \qquad y_2 = 1 \qquad y_3 = 1$$

be given. In this problem, we want to find the best least squares fit for

$$y = x^T M x$$

where M is a symmetric matrix of the form

$$M = \begin{bmatrix} a & b \\ b & a \end{bmatrix}.$$

(a) Find the normal equations.

(b) Find the least squares solution.

4 Eigenvalues/eigenvectors $10 + 15 = 25$ pts

Let a be a nonzero real number.

(a) Show that the matrix

$$\begin{bmatrix} a & a & a \\ a & a & a \\ a & a & a \end{bmatrix}$$

is diagonalizable.

(b) Find a nonsingular matrix X and a diagonal matrix D such that $A = XDX^{-1}$.

10 pts free